

Multigluon Tree Amplitudes and Self-Duality Equation

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Abstract

A generating function for a class of multigluon amplitudes is constructed as a particular solution of the self-duality equation.

1. Recently, there has been developed an interesting activity based on the fact that a soliton solution in a generic massive field theory happens to be a generating function for three multiparticle amplitudes with most of the particles at threshold (see reviews [1]). The corresponding technique essentially improves the perturbative expansion for the threshold multiparticle processes [2], [3] and sometimes allows to take into account all loop corrections [4]. Another interesting result having come from that approach is very nontrivial nullifications among the tree threshold amplitudes [5]. The nullification was shown to survive in some theories at the quantum level [6]. In a particular model it was explained by a hidden conservation law [7] and it was given a general interpretation in terms of the algebro-geometric approach to solitonic equations in [8].

On the other hand side, there has been developed an involved technology for efficient calculating the multigluon amplitudes in Yang-Mills (YM) theory, basically, at tree level (reviewed in [9]) and at one-loop level (reviewed, e.g., in [10]). One of the main ingredients is the spinor helicity formalism [11]. Interestingly, the multigluon amplitudes have a counterpart of the nullification mentioned above. Namely, the tree amplitudes with all and with all but one external gluons having identical helicities are zero [9].

This note is aimed to bring these two seemingly different activities together by demonstrating that a generating function for the multigluon amplitudes can be obtained by solving the self-duality (SD) equations.

I consider here only the SU(2) case and use the simple 't Hooft ansatz to deal with the SD equation. The SU(N) case will be considered separately with use of the twistors for constructing the appropriate solution, which naturally lead to the spinor helicity nitti gritti [11].

2. The starting point is the following version of the Lehmann-Symanzik-Zimmerman formula (see also [12]):

$$\left(\frac{\delta S_{tree}(\{a, a^*\})}{\delta a_\lambda^j(p)} \right)_c = i \int d^4x \frac{1}{2\omega_p} e^{-ipx} \varepsilon_\mu^\lambda \cdot (\partial^2 \delta_{\mu\nu} - \partial_\mu \partial_\nu) A_\nu^j(x, \{a, a^*\}) \quad (1)$$

where $px = \omega_p t - \vec{p}\vec{x}$, $S(\{a, a^*\})$ is the so-called normal symbol of S -matrix, the subscript "tree" indicates the tree approximation, the subscript "c" means that only connected part is included in the amplitudes, $a_\lambda^j(p)$ ($a_\lambda^{*j}(p)$) stands for the symbol of annihilation (creation) operator of the gluon state with momentum p , color index j and a polarization λ , ε_μ^λ is a four-vector defining the polarization λ ($\varepsilon_\mu^\lambda p^\mu = 0$) and A_ν^j is a solution of the classical equation of motion. The $\{a, a^*\}$ -dependence of A_ν^j comes via the Feynman-type boundary asymptotic condition:

$$A_\nu^j(x, \{a, a^*\}) = A_\nu^{0j}(x, \{a, a^*\}) + O(g) \quad (2)$$

where g is a coupling constant and A^0 is a solution of the free (without nonlinearity) equation of motion,

$$A_\nu^{0j} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (a_\lambda^j(k) \varepsilon_\nu^\lambda e^{-ikx} + a_\lambda^{*j}(k) \bar{\varepsilon}_\nu^\lambda e^{ikx}) . \quad (3)$$

Some comments are in order here:

1) the condition (2) fixes the solution of the classical equation of motion uniquely, provided the perturbation theory is well-defined, and due to this uniqueness a solution matching the condition (3) is the appropriate one;

2) the power expansion of the formula (1) in a, a^* produces the multigluon amplitudes (with one punctured gluon) in the obvious way;

3) the A^0 in (3) need not include all the a and a^* in the theory, in this case the formula (1) generates a subclass of the tree amplitudes, corresponding to the a, a^* kept;

4) the punctured particle in (1) (corresponding to $a_\lambda^j(p)$) need not enter the A^0 in (3), moreover, it need not even be on-shell, in the latter case the formula (1) generates rather formfactors than the amplitudes and is not gauge invariant;

5) when the punctured particle is taken on shell, the amplitude is nonzero only if the solution A in (1) is sufficiently singular;

6) to have a possibility to keep two particles off shell or off the asymptotic condition (3) one should take the variation of (1) with respect to another a , the A in (1) is then substituted by $\frac{\delta A}{\delta a}$ which obviously obeys the variation of the classical equation of motion (at this point is essential that only connected part of the amplitudes are considered).

One can not proceed further in all generality as the YM equation is not integrable and one cannot find its solution for an arbitrary boundary condition (3). However, according to the comment 3), one can keep such a subset of a 's and a^* 's in (3) that the corresponding A obeys simpler equation than the YM one. In the massive theory one proceeded keeping only space-uniform subset of a 's and a^* 's, the classical equation of motion reducing to an ordinary differential equation. That was the basic idea behind the threshold amplitudes activity mentioned in the introduction. Here I take A^0 such that A obeys the SD equation. To be rigorous, note that in Minkovsky space (where we are) the $*$ -operator squares to -1 and the SD equation says that the curvature from is a $*$ -eigenform with eigenvalue $i = \sqrt{-1}$, but it is not crucial, as after the reduction to a subset of a 's and a^* 's in (3) the A^0 and A need not be real. The appropriate solutions can actually be obtained by the Wick's analytical continuation. Using the (analytically continued) 't Hooft ansatz

$$A = \frac{i}{g} \bar{\Sigma}_{\mu\nu} \partial_\nu \ln \Phi dx^\mu \quad (4)$$

the SD equation is known to reduce to the following equation on Φ :

$$\frac{1}{\Phi} \square \Phi = 0 \quad (5)$$

(∂_ν and dx^μ include $i = \sqrt{-1}$ where it is necessary according to the Wick's rotation rules, $\bar{\Sigma}_{\mu\nu}$ are the 't Hooft matrixes, $\bar{\Sigma}_{\mu\nu} = -\bar{\Sigma}_{\nu\mu}$, $\bar{\Sigma}_{a0} = -\frac{\sigma_a}{2}$, $\bar{\Sigma}_{ab} = \varepsilon^{iab} \frac{\sigma^i}{2}$, σ^i are the Pauli matrixes). For the current purposes I take the solution

$$\Phi = 1 + g \int \frac{d^3 q}{(2\pi)^3 2q_0} (\alpha(q) e^{-iqx} + \alpha^*(q) e^{iqx}) \quad (6)$$

with $q_0^2 - (\vec{q})^2 = 0$. Substituting Φ into (4) and expanding in g up to first order relates α, α^* to a, a^* . At this point I need to describe special gluon states arising in the asymptotics of the solution (6), (4). This will define the subclass of the multigluon amplitudes for which this solution plays the role of generating function.

Consider right triple of orthogonal 3-vectors, $(\vec{n}_1, \vec{n}_2, \vec{q})$, \vec{q} being the spatial part of q in (6). Introduce circular polarization $\varepsilon^-(n, q) = (0, \vec{n}_1 - i\vec{n}_2)$. Introduce also matrixes $\sigma_{n_i}(nq) = \sigma^k n_i^k$ and a color orientation $\sigma^+(n, q) = \sigma_{n_1} + i\sigma_{n_2}$. That are these gluon states, with the circular polarization $\vec{\varepsilon}^-(n, q)$ and the color polarization $\sigma^+(n, q)$ which will appear in the asymptotic states. $a_{+,-}(q)$ and $a_{+,-}^*(q)$ will stand for the corresponding annihilation and creation symbols (notice that the states are independent of the choice of (\vec{n}_1, \vec{n}_2) at given q and the time direction. The parameters α, α^* from (6) read then as follows:

$$\begin{aligned}\alpha(q) &= \frac{2a_{+,-}(q)}{iq_0} \\ \alpha^*(q) &= \frac{2ia_{+,-}^*(q)}{q_0}\end{aligned}\tag{7}$$

The transversal part of the solution A (which only contributes in (1)) is

$$\begin{aligned}A_k^T &= \frac{1}{\Phi} \int \frac{d^3q}{(2\pi)^3 2q_0} [a_{+,-}(q) \sigma^+(n, q) \varepsilon_k^-(u, q) e^{-iqx} + \\ &+ a_{+,-}^*(q) \sigma^+(n, q) \varepsilon_k^-(n, q) e^{iqx}]\end{aligned}\tag{8}$$

One immediately sees that the solution (8) does not have any singularity which could give a nonzero amplitude when the punctured gluon is on-shell (independently of its polarization and color orientation). This is the nullification cited above. When the punctured gluon is off-shell, the formula (8) substituted into (1) generates nonzero formfactors (which are not gauge invariant, of course). As was said in comment 5), to find the amplitudes with two arbitrary polarized and color oriented gluon and arbitrary number of the specially polarized and oriented gluons one needs to solve the variation of the YM equation in the background of $A(x, \{a, a^*\})$. This will be done elsewhere.

3. I would like to indicate what developments are possible on the basis of exposed in this note.

1) $SU(N)$ case allows similar construction. In that general case the spinor helicity formalism comes naturally via the twistor description of solutions of SD equation.

2) The variation of the YM equation can be solved in the background of $A(x, \{a, a^*\})$ (even in $SU(N)$ case) which will allow to construct the generating function for the processes with two arbitrarily polarized gluons and arbitrary number of the specially polarized ones. Actually, the recursive relations used in [13], [14] to find this type of amplitudes must be just a perturbative expansion of the variation of YM.

3) Constructing the Green function in the background of $A(x, \{aa^*\})$ will allow to develop a perturbation theory for any processes beyond the tree

approximation including arbitrary number of special gluons, similar to [2], [3] in the threshold amplitudes activity. I would like to recall that using this kind of technique the authors of [4] (see also [15]) managed to collect all loops contributions in the leading term of the threshold amplitude when the number of particle is of the order of inverse coupling constant.

4) The nullification of the amplitudes with the specially polarized gluons was explained by an effective supersymmetry Ward identities [16] (the idea was that the normal YM can be completed up to the supersymmetric one, while the superpartners do not contribute at the tree level). A similar interpretation of the nullification in the threshold amplitudes could easily be possible.

5) In [17] a general approach based on the Whitham technique was developed to treat arbitrary next-to-threshold amplitudes. By the way, the effective theory was nicely formulated in terms of the modular geometry and happened to be solvable at the classical level. I hope that an analogous construction is possible in the present case. The corresponding effective theory would describe processes including the specially polarized gluons with arbitrary momenta and arbitrary polarized soft gluons.

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